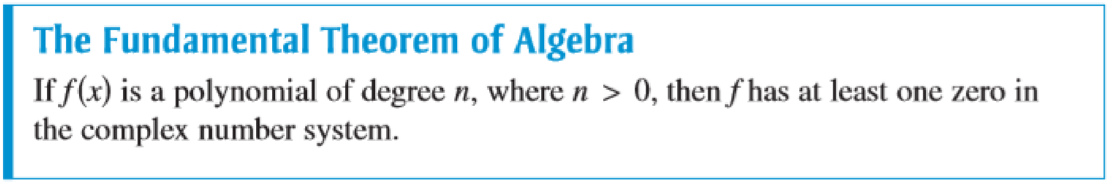
Section 2.5 A

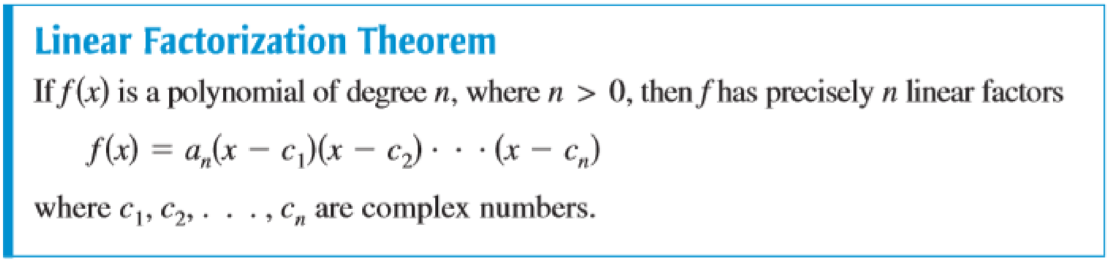
Synthetic Division Rational Roots

- Use the Fundamental Theorem of Algebra to determine the number of zeros of polynomial functions.
- Find rational zeros of polynomial functions.
- Find zeros of polynomials by factoring.



The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.



Linear Factorization Theorem

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where $c_1, c_2, . . . , c_n$ are complex numbers.

Note that the Fundamental Theorem of Algebra and the Linear Factorization Theorem tell you only that the zeros or factors of a polynomial exist, not how to find them.

How many zeros and what are they?

$f(x) = x' - 2$ $\quad 0 = x - 2$

zero: 1 $\quad x = 2$

$f(x) = x^{\textcircled{2}} - 6x + 9$ $\quad x^2 - 6x + 9 = 0$

zeros: 2

$(x-3)(x-3) = 0$

$x = 3$ multiplicity 2

$f(x) = x^{\textcircled{3}} + 4x$ $\quad x^3 + 4x = 0$ $\quad x^2 + 4 = 0$

zeros: 3

$x(x^2+4) = 0$ $\quad x^2 = -4$

$x = 0 \quad x = \pm 2i$ $\quad x = \pm\sqrt{-4}$

$f(x) = x^{\textcircled{4}} - 1$ $\quad x^4 - 1 = 0$

zeros: 4

$(x^2+1)(x^2-1) = 0$

$(x+i)(x-i)(x+1)(x-1) = 0$

$x = \pm i \quad x = \pm 1$

The Rational Zero Test

If the polynomial $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdot\cdot\cdot + a_2x^2 + a_1x + a_0$ has *integer* coefficients, every rational zero of f has the form

$$\text{Rational zero} = \frac{p}{q}$$

where p and q have no common factors other than 1, and

p = a factor of the constant term a_0

q = a factor of the leading coefficient a_n.

Find the rational zeros. $\pm \frac{1}{1}$

$f(x) = \overset{q}{1}x^3 + x + \overset{p}{1}$

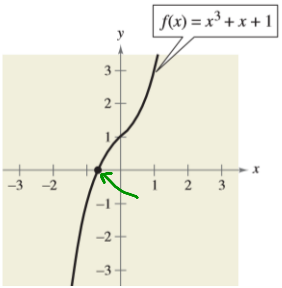
± 1

$$\begin{array}{r|rrrr} 1 & 1 & 0 & 1 & 1 \\ & \downarrow & 1 & 1 & 2 \\ \hline & 1 & 1 & 2 & 3 \end{array}$$

upper bound

$$\begin{array}{r|rrrr} -1 & 1 & 0 & 1 & 1 \\ & \downarrow & -1 & 1 & -2 \\ \hline & 1 & -1 & 2 & -1 \end{array}$$

lower bound



$x = .6823$

+ 2 imaginary zero

Find the rational zeros of $\frac{p}{q}$

$f(x) = \overset{Q}{1}x^4 - x^3 + x^2 - 3x - \overset{P}{6}$

Rational: $X = -1, 2$

$1\cdot6$
$2\cdot3$

$\pm \frac{1, 2, 3, 6}{1}$

$\pm 1, \pm 2, \pm 3, \pm 6$

$$\begin{array}{r|rrrrr}
-1 & 1 & -1 & 1 & -3 & -6 \\
 & \downarrow & -1 & 2 & -3 & 6 \\
\hline
2 & 1 & -2 & 3 & -6 & 0 \\
 & \downarrow & 2 & 0 & 6 & \\
\hline
 & 1 & 0 & 3 & 0 &
\end{array}$$

$X^2 + 3 = 0$

$X^2 = -3$

$X = \pm\sqrt{-3}$

$X = \pm i\sqrt{3}$

Find all of the real zeros.

Helpful hints:

1) rational zero test: p/q

2) intermediate value theorem: (change in signs of the remainder)

3) upper and lower bound

4) use graphing calculator

Upper and Lower Bound Rules

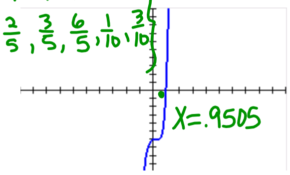
Let $f(x)$ be a polynomial with real coefficients and a positive leading coefficient. Suppose $f(x)$ is divided by $x - c$, using synthetic division.

1. If $c > 0$ and each number in the last row is either positive or zero, c is an **upper bound** for the real zeros of f.
2. If $c < 0$ and the numbers in the last row are alternately positive and negative (zero entries count as positive or negative), c is a **lower bound** for the real zeros of f.

Prove that all of the real zeros lie between the interval $[\overset{\text{lower}}{0}, \overset{\text{upper}}{1}]$

What are the potential rational zeros? $\frac{p}{q}$ $\pm \frac{1,2,3,6}{1,2,5,10}$

$f(x) = \overset{Q}{10}x^5 - 3x^2 + x - \overset{P}{6}$ $\pm\left\{1,2,3,6,\frac{1}{2},\frac{3}{2},\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{6}{5},\frac{1}{10},\frac{3}{10}\right\}$



1	10	0	0	-3	1	-6
	↓	10	10	10	7	8
	10	10	10	7	8	2

upper bound

0	10	0	0	-3	1	-6
	↓	0	0	0	0	0
	10	0	0	-3	1	-6
	+	-	+			

lower bound

$\frac{3}{5}$	10	0	0	-3	1	-6
	↓	6	3.6	2.16	-.504	.2976
	10	6	3.6	-.84	.496	-5.7024

Find the real zeros:

$$f(x) = x^3 - 15x^2 + 75x - 125$$

$\frac{p}{q} = \pm\ 1, 5, 25, 125$

$x = 5$ mult. (3)

5	1	-15	75	-125
	↓	5	-50	125
	1	-10	25	0

$x^2 - 10x + 25 = 0$

$(x-5)(x-5) = 0$

$x = 5, 5$

Find the real zeros:

$X = -3, 1, \frac{1}{2}$

$$f(x) = 2x^3 + 3x^2 - 8x + 3$$

$$\frac{p}{q} = \pm \frac{1,3}{1,2} = \pm\left\{1, 3, \frac{1}{2}, \frac{3}{2}\right\}$$

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & 3 & -8 & 3 \\ & \downarrow & 1 & 2 & -3 \\ \hline & 2 & 4 & -6 & 0 \end{array}$$

$$2x^2 + 4x - 6 = 0$$

$$2(x^2 + 2x - 3) = 0$$

$$2(x+3)(x-1) = 0$$

$$X = -3, 1$$

Find the real zeros:

$$-10x^3 + 15x^2 + 16x - 12 = 0$$

$$\frac{p}{q} = \pm \frac{1,2,3,4,6,12}{1,2,5,10}$$

$$\begin{array}{r|rrrr} 2 & -10 & 15 & 16 & -12 \\ & \downarrow & -20 & -10 & 12 \\ \hline & -10 & -5 & 6 & 0 \end{array}$$

$$-10x^2 - 5x + 6 = 0$$

$$-(10x^2 + 5x - 6) = 0$$

$$X = \frac{-5 \pm \sqrt{25 - 4(10)(-6)}}{2(10)}$$

$$X = \frac{-5 \pm \sqrt{265}}{20}$$

Section 2.5A

Pg. 162-165: #9-12, 17, 18, 22, 27, 29-39 odd